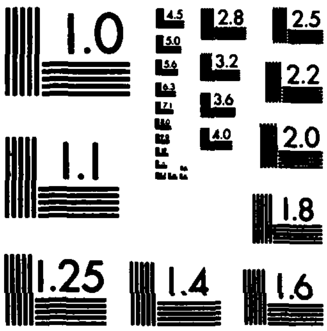
AD-A135 023 NYQUIST FREQUENCY(U) MASSACHUSETTS INST OF TECH CAMBRIDGE STATISTICS CENTER Y C YAO SEP 83 TR-32-ONR N00014-75-C-0555 1/1

UNCLASSIFIED F/G 9/4 NL

END
FILMED
1-84
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

(13)

STATISTICS CENTER

Massachusetts Institute of Technology

77 Massachusetts Avenue, Rm. E40-111, Cambridge, Massachusetts 02139 (617) 253-8722

NYQUIST FREQUENCY

BY

YI-CHING YAO
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

TECHNICAL REPORT NO. ONR 32

SEPTEMBER 1983

PREPARED UNDER CONTRACT

N00014-75-C-0555 (NR-609-001)

FOR THE OFFICE OF NAVAL RESEARCH

Nyquist Frequency

Abstract

Nyquist frequency is related to aliasing and the sampling theorem. This report explains the relationship among these terms.

Key words: Aliasing, Nyquist frequency, spectrum analysis, the sampling theorem

pi/delta

The Nyquist frequency is half the sampling frequency when a continuous-time function is sampled at equally spaced time points. That is, the Nyquist frequency is π/Δ (in radians per unit time) where delta Δ is the time interval between two successive sampled data. This report discusses the ~~In this article, we discuss~~ the basic ideas of the Nyquist frequency and explains its relationship between a relevant and very ~~important theorem - the sampling theorem.~~ aliasing and the sampling theorem.

In many applications involving processing a continuous-time signal, it is often preferable to convert the continuous-time signal to a discrete-time signal since discrete-time signal processing can be implemented with a digital computer. It is important to examine whether the discrete-time signal preserves all the information in the original continuous-time signal. We first consider the case that the signal $x(t)$ is a real-valued function. Assume that its Fourier transform $X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-i\omega t)\,dt$ exists and that $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(i\omega t)\,d\omega$ for all t. The signal $x(t)$ is sampled at $t = n\Delta$, $n = \ldots, -1, 0, 1, \ldots$. We are interested in interpolating $x(t)$ from its samples $x(n\Delta)$. A natural question arises: Under what conditions can $x(t)$ be perfectly reconstructed from $x(n\Delta)$? The samples $x(n\Delta)$ can be related to $X(\omega)$ as follows.

$$x(n\Delta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(i\omega n\Delta)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/\Delta}^{\pi/\Delta}\Big[\sum_{k=-\infty}^{\infty} X(\omega + 2\pi k/\Delta)\Big]\exp(i\omega n\Delta)\,d\omega$$

Therefore, $X_d(\omega) = \Sigma X(\omega + 2\pi k/\Delta)$ $(|\omega| < \pi/\Delta)$ is the discrete Fourier transform of the sequence $x(n\Delta)$. Obviously, $X_d(\cdot)$ is obtained by folding $X(\cdot)$ every π/Δ radians per unit time. (Here we identify ω with $-\omega$.) This frequency π/Δ is the Nyquist frequency, and hence also called the folding frequency. It is easy to see that $X(\cdot)$ is not uniquely determined by $X_d(\cdot)$. In other words, some sinusoidal components of different frequencies (e.g. $2\pi k/\Delta \pm \omega_o$, $k = \ldots,-1,0,1,\ldots$) in $x(t)$ cannot be distinguished from one another by the observations $x(n\Delta)$. This is called aliasing. Aliasing is the effect of under-sampling. This effect is the principle on which the stroboscopic effect is based [6, Section 8.3].

When $x(t)$ is a band-limited signal with $X(\omega) = 0$ for $|\omega| > \omega_M$, $X_d(\omega)$ is identical to $X(\omega)$ (i.e. no aliasing) if $\omega_M < \pi/\Delta$. In other words, from the uniqueness property of Fourier transform, $x(t)$ is uniquely determined by its samples $x(n\Delta)$ under the condition that $X(\omega) = 0$ for $|\omega| > \pi/\Delta$. This is usually called the (Shannon) sampling theorem on information theory [5]. From the sampling theorem, if we sample the signal $x(t)$ at a rate at least twice the highest frequency in $x(t)$, then $x(t)$ can be completely recovered from the samples. This sampling rate (twice the highest frequency in $x(t)$) is commonly referred to as the Nyquist rate. Actually $x(t)$ can be explicitly written, in terms of $x(n\Delta)$, as

$$x(t) = \sum_{n=-\infty}^{\infty} x(n\Delta) \frac{\sin \pi\{(t/\Delta)-n\}}{\pi\{(t/\Delta) - n\}}$$

It should be noted that band-limited signals are generally not realizable physically, for $x(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} X(\omega)\exp(i\omega t)d\omega$ is analytic in t, as a complex variable, and therefore cannot vanish for all $t < -T$ for arbitrarily large T. Therefore, aliasing is inevitable in practice. A discussion on error bounds for aliasing can be found in [5].

In some applications, the signal x(t) is assumed to be bandpass, i.e. there exist $0 \leq \omega_0 < \omega_1$ such that $X(\omega) = 0$ outside the intervals $[\omega_0,\omega_1]$ and $[-\omega_1,-\omega_0]$. The sampling theorem says that x(t) can be recovered from equally spaced sampling at a rate of $2\omega_1$. Actually, this rate $2\omega_1$ is too conservative. It has been shown [4, Section 8.5] that a sampling rate of $2\omega_1/\nu$ is enough to recover x(t) where ν is the largest integer not beyond $\omega_1/(\omega_1-\omega_0)$.

The sampling theorem has been generalized to many situations such as random signals. When x(t) $(-\infty < t < \infty)$ is a wide-sense stationary stochastic process, possessing a spectral density which vanishes outside the interval $[-\pi/\Delta,\pi/\Delta]$, Balakrishnan showed [1] that x(t) has the representation

$$x(t) = \lim_{N\to\infty} \sum_{n=-N}^{N} x(n\Delta) \frac{\sin \pi\{(t/\Delta)-n\}}{\pi\{(t/\Delta)-n\}}$$

for every t, where lim stands for limit in the mean square. Gardner [3] derived a similar result for non-stationary stochastic processes. Obviously, the Nyquist frequency and

Nyquist rate can be similarly defined in the random signal case.

Blackman and Tukey [2, Section 12] provided good interpretations on aliasing. Jerri [5] gave an excellent review of the sampling theorem and its various extensions and applications. He discussed topics such as unequally spaced sampling, higher-dimensional functions, nonband-limited functions and error bounds for the truncation, aliasing, and jitter. A very exhaustive bibliography can be found therein.

References

[1] Balakrishnan, A.V. (1957). *IRE Trans. Inform. Theory*, IT-3, pp. 143-146.

[2] Blackman, R.B. and Tukey, J.W. (1958). *The Measurement of Power Spectra*, Dover, New York.

[3] Gardner, W.A. (1972). *IEEE Trans. Inform. Theory*, IT-18, pp. 808-809.

[4] Gregg, W.D. (1977). *Analog and Digital Communication*, John Wiley, New York.

[5] Jerri, A.J. (1977). *Proc. IEEE*, 65, pp. 1565-1596.

[6] Oppenheim, A.V., Willsky, A.S. and Young, I.T. (1983). *Signals and Systems*, Prentice-Hall, Englewood Cliffs, N.J.

(INFORMATION THEORY
SAMPLING
SPECTRUM ANALYSIS)

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 32	2. GOVT ACCESSION NO.: AD-A135 023
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): Nyquist Frequency	5. TYPE OF REPORT & PERIOD COVERED: Technical Report
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Yi-Ching Yao	8. CONTRACT OR GRANT NUMBER(s): N00014-75-C-0555
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Massachusetts Institute of Technology, Statistics Center, Cambridge, MA 02139	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: (NR-609-001)
11. CONTROLLING OFFICE NAME AND ADDRESS: Office of Naval Research, Statistics and Probability Code 436, Arlington, VA 22217	12. REPORT DATE: September 1983
	13. NUMBER OF PAGES: 5
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report)
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

This document has been approved for public release and sale; its distribution is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Aliasing, Nyquist frequency, spectrum analysis, the sampling theorem.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Nyquist frequency is related to aliasing and the sampling theorem. This report explains the relationship among these terms.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

END

FILMED

1-84

DTIC